Basic Trigonometric Identities

We have already seen the following identities:

Reciprocal Identities:

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Tangent and Cotanget in Terms of Sine and Cosine:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

The Fundamental Identity:

$$\sin^2 x + \cos^2 x = 1$$

If we divide each term of the fundamental identity by $\sin^2 x$ or $\cos^2 x$, we can derive two more identities. These are called Pythagorean Identities because they are related to the Pythagorean Theorem:

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$1 + \cot^2 x = \csc^2 x$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

Simplifying Expressions

We can use the identities above to simplify trigonometric expressions. One of the most common strategies is to start by rewriting the expression in terms of sines and/or cosines, then simplify from there.

Examples:

$$\frac{\tan x}{\sec x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1}$$

$$= \boxed{\sin x}$$

$$\sin x + \cot x \cos x =$$

$$\sin x + \left(\frac{\cos x}{\sin x}\right)\cos x =$$

$$\sin x + \frac{\cos^2 x}{\sin x} =$$

$$\left(\frac{\sin x}{\sin x}\right)\sin x + \frac{\cos^2 x}{\sin x} =$$

$$\frac{\sin^2 x + \cos^2 x}{\sin x} = \frac{1}{\sin x}$$

$$= \boxed{\csc x}$$

$$\frac{\tan x \csc x}{\sec x} =$$

$$\frac{\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right)}{\left(\frac{1}{\cos x}\right)} =$$

$$\frac{\left(\frac{1}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)} = \boxed{1}$$

Writing One Function in Terms of Another

We sometimes want to write one identity in terms of another. For example, we could write the cosine function in terms of the sine by solving the fundamental identity to get $\cos x = \pm\sqrt{1-\sin^2 x}$.

Examples:

Write the tangent in terms of the sine.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \frac{\sin x}{\pm\sqrt{1-\sin^2 x}}$$

Write the cotangent in terms of the cosine.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1-\cos^2 x \qquad \sin x = \pm\sqrt{1-\cos^2 x}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{\cos x}{\pm\sqrt{1-\cos^2 x}}$$

Using Identities to Find Function Values

If we know the value of one trigonometric function for an angle, we can use trigonometric identities to find the values of the other five functions.

Examples:

$\cos\alpha$ & $\sec\alpha$ are positive

If $\tan\alpha = -2/3$ and α is in quadrant IV, find the values of the remaining five trigonometric functions.

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{-2/3} = -3/2$$

$$\tan^2\alpha + 1 = \sec^2\alpha$$

$$(-2/3)^2 + 1 = \sec^2\alpha$$

$$4/9 + 1 = \sec^2\alpha$$

$$13/9 = \sec^2\alpha$$

$$\sec\alpha = \sqrt{13/9} = \sqrt{13}/3$$

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{\sqrt{13}/3} = \frac{3}{\sqrt{13}}$$

$$1+\cot^2\alpha = \csc^2\alpha$$

$$1 + (-3/2)^2 = \csc^2\alpha$$

$$1 + 9/4 = \csc^2\alpha$$

$$13/4 = \csc^2\alpha$$

$$\csc\alpha = -\sqrt{13/4} = -\sqrt{13}/2$$

$$\sin\alpha = \frac{1}{\csc\alpha} = \frac{1}{-\sqrt{13}/2} = -\frac{2}{\sqrt{13}}$$

Odd and Even Identities

An odd function is a function for which $f(-x) = -f(x)$, and an even function is one for which $f(-x) = f(x)$. The graph of an odd function is symmetric about the origin, and the graph of an even function is symmetric about the *y*-axis. Of the six trigonometric functions, the cosine and secant are even, and the others are odd.

Odd:	$\sin(-x) = -\sin x$	$\csc(-x) = -\csc x$
	$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$
Even:	$\cos(-x) = \cos x$	$\sec(-x) = \sec x$

Examples:

$\csc(-x)\tan(-x) =$

$$(-\csc x)(-\tan x) =$$

$$\csc x \tan x =$$

$$\frac{1}{\sin x}\cdot\frac{\sin x}{\cos x} =$$

$$\frac{1}{\cos x} = \boxed{\sec x}$$

$\dfrac{1}{1+\cos(-x)} + \dfrac{1}{1-\cos x}$ $= \dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x}$

$$\left(\frac{1-\cos x}{1-\cos x}\right)\left(\frac{1}{1+\cos x}\right) + \left(\frac{1}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right) =$$

$$\frac{1-\cos x + 1 + \cos x}{(1-\cos x)(1+\cos x)} = \frac{2}{1-\cos^2 x}$$

$$= \frac{2}{\sin^2 x} = \boxed{2\csc^2 x}$$

Verifying Identities

It is often necessary to determine whether two expressions are equivalent to each other. We can use the approaches from the previous section to verify whether equations are identities.

Multiplying and Factoring Polynomials Involving Trigonometric Functions

We must often multiply binomials or factor trinomials involving trigonometric functions when we verify identity.

Examples:

Multiply $(1+\tan x)(1-\tan x) =$

$$1-\tan x+\tan x-\tan^2 x$$

$$=\boxed{1-\tan^2 x}$$

Multiply $(2\sin x+1)^2 = (2\sin x+1)(2\sin x+1)$

$$= 4\sin^2 x+2\sin x+2\sin x+1$$

$$=\boxed{4\sin^2 x+4\sin x+1}$$

Factor $\sec^2 x-\tan^2 x =$

$$\boxed{(\sec x+\tan x)(\sec x-\tan x)}$$

Factor $\sin^2 x+\sin x-2 =$

$$\boxed{(\sin x+2)(\sin x-1)}$$

$-2 \mid 1$
$2,-1 \mid 1$
$-2,1 \mid -1$

A General Strategy for Verifying Identities

1. Work on the more complicated side first.
2. Rewrite the side you are working with in terms of sines and cosines only.
3. Multiply the numerator and denominator of one rational expression by either the numerator or denominator of the other.
4. Write a single rational expression as a sum of two rational expressions.
5. Combine a sum of two rational expressions into a single rational expression.
6. If both sides simplify to a third expression, then the equation is an identity.

Examples:

Verify that $1+\sec x\sin x\tan x=\sec^2 x$ is an identity.

$$1+\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{1}\right)\left(\frac{\sin x}{\cos x}\right) = \sec^2 x$$

$$1+\frac{\sin^2 x}{\cos^2 x} = \sec^2 x$$

$$1+\tan^2 x = \sec^2 x$$

$$\sec^2 x = \sec^2 x \quad \checkmark$$

Prove that $\frac{\cos\alpha}{1-\sin\alpha}=\frac{1+\sin\alpha}{\cos\alpha}$ is an identity.

$$\frac{\cos\alpha}{1-\sin\alpha} = \left(\frac{1+\sin\alpha}{\cos\alpha}\right)\left(\frac{1-\sin\alpha}{1-\sin\alpha}\right)$$

$$\frac{\cos\alpha}{1-\sin\alpha} = \frac{1-\sin^2\alpha}{\cos\alpha(1-\sin\alpha)}$$

$$\frac{\cos\alpha}{1-\sin\alpha} = \frac{\cos^2\alpha}{\cos\alpha(1-\sin\alpha)}$$

$$\frac{\cos\alpha}{1-\sin\alpha} = \frac{\cos\alpha}{1-\sin\alpha} \quad \checkmark$$

Prove that $\dfrac{\csc x - \sin x}{\sin x} = \cot^2 x$ is an identity.

$$\frac{\frac{1}{\sin x} - \sin x}{\sin x} = \cot^2 x$$

$$\frac{\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}}{\sin x} = \cot^2 x$$

$$\frac{\frac{1-\sin^2 x}{\sin x}}{\sin x} = \cot^2 x$$

$$\frac{\cos^2 x}{\sin x} \cdot \frac{1}{\sin x} = \cot^2 x$$

$$\frac{\cos^2 x}{\sin^2 x} = \cot^2 x$$

$$\cot^2 x = \cot^2 x$$

Prove that $-2\cot^2 x = \dfrac{1}{1-\sec x} + \dfrac{1}{1+\sec x}$ is an identity.

$$-2\cot^2 x = \left(\frac{1}{1-\sec x}\right)\left(\frac{1+\sec x}{1+\sec x}\right) + \left(\frac{1}{1+\sec x}\right)\left(\frac{1-\sec x}{1-\sec x}\right)$$

$$-2\cot^2 x = \frac{1+\sec x + 1 - \sec x}{1-\sec^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 - \sec^2 x = -\tan^2 x$$

$$-2\cot^2 x = \frac{2}{-\tan^2 x}$$

$$-2\cot^2 x = -2\cot^2 x$$

Prove that $\dfrac{1-\sin^2 t}{1-\csc(-t)} = \dfrac{1+\sin(-t)}{\csc t}$ is an identity.

$$\frac{1-\sin^2 t}{1+\csc t} = \frac{1-\sin t}{\csc t}$$

$$\frac{(1-\sin t)(1+\sin t)}{1+\frac{1}{\sin t}} = \frac{1-\sin t}{1/\sin t}$$

$$\frac{(1-\sin t)(1+\sin t)}{\frac{\sin t}{\sin t} + \frac{1}{\sin t}} = \sin t(1-\sin t)$$

$$\frac{(1-\sin t)(1+\sin t)}{\frac{\sin t + 1}{\sin t}} = \sin t(1-\sin t)$$

$$(1-\sin t)\cancel{(1+\sin t)}\left(\frac{\sin t}{\cancel{(\sin t+1)}}\right) = \sin t(1-\sin t)$$

$$\sin t\,(1-\sin t) = \sin t(1-\sin t)$$

Show that $\dfrac{1-\cos^2(-t)}{\sin(-t)} = \tan(-t)\cos(-t)$ is an identity.

$$\frac{1-\cos^2 t}{-\sin t} = (-\tan t)(\cos t)$$

$$\frac{\sin^2 t}{-\sin t} = \left(\frac{-\sin t}{\cos t}\right)(\cos t)$$

$$-\sin t = -\sin t$$

Sum and Difference Identities for Cosine

Often, an angle can be expressed as a sum or difference of two angles for which we know the exact values of the trigonometric functions. We can use sum and difference identities to find the exact values of the trigonometric functions of our angle of interest.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Mnemonic: "Cosine changes the silly sign."

Examples: Write the following angles as sums or differences of two other angles whose trigonometric functions can be calculated exactly:

$105^\circ = 45^\circ+60^\circ$, $135^\circ-30^\circ$

$15^\circ = 60^\circ-45^\circ$, $45^\circ-30^\circ$

$345^\circ = 300^\circ+45^\circ$, $315^\circ+30^\circ$

$195^\circ = 150^\circ+45^\circ$, $135^\circ+60^\circ$

$\frac{7\pi}{12} = \frac{\pi}{4}+\frac{\pi}{3}$, $\frac{3\pi}{4}-\frac{\pi}{6}$

$\frac{\pi}{12} = \frac{\pi}{3}-\frac{\pi}{4}$, $\frac{\pi}{4}-\frac{\pi}{6}$

$-\frac{\pi}{12} = \frac{\pi}{4}-\frac{\pi}{3}$, $\frac{\pi}{6}-\frac{\pi}{4}$

$\frac{5\pi}{12} = \frac{\pi}{6}+\frac{\pi}{4}$, $\frac{3\pi}{4}-\frac{\pi}{3}$

Find the exact value of $\cos(75^\circ)$.

$\cos(30^\circ+45^\circ)=$

$\cos 30^\circ\cos 45^\circ-\sin 30^\circ\sin 45^\circ$

$\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$

Find the exact value of $\cos\left(\frac{\pi}{12}\right)$.

$\cos(\pi/3-\pi/4)=$

$\cos(\pi/3)\cos(\pi/4)+\sin(\pi/3)\sin(\pi/4)$

$(1/2)(\sqrt{2}/2)+(\sqrt{3}/2)(\sqrt{2}/2)$

$\sqrt{2}/4+\sqrt{6}/4=\boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

Use appropriate identities to simplify each expression:

$\cos 49^\circ\cos 4^\circ+\sin 49^\circ\sin 4^\circ$

$=\cos(49^\circ-4^\circ)=\cos(45^\circ)=\boxed{\frac{\sqrt{2}}{2}}$

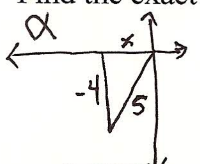
$\cos(x)\cos(-5x)+\sin(-x)\sin(5x)$

$=\cos(x)\cos(5x)+(-\sin(x))\sin(5x)$

$=\cos(x)\cos(5x)-\sin(x)\sin(5x)$

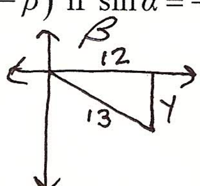
$=\cos(x+5x)=\boxed{\cos(6x)}$

Find the exact value of $\cos(\alpha-\beta)$ if $\sin\alpha=-4/5$ and $\cos\beta=12/13$. α is in Quadrant III. β is in Quadrant IV.



$\cos\alpha=\frac{-3}{5}$

$x=\sqrt{5^2-4^2}=-3$



$\sin\beta=-5/13$

$y=-\sqrt{13^2-12^2}=-5$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=(-3/5)(12/13)+(-4/5)(-5/13)$

$=-36/65+20/65=\boxed{\frac{-16}{65}}$

Cofunction Identities

Sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. Each function can be related to its cofunction by a simple cofunction identity.

$$\sin\left(\frac{\pi}{2}-u\right)=\cos u \qquad \cos\left(\frac{\pi}{2}-u\right)=\sin u$$
$$\tan\left(\frac{\pi}{2}-u\right)=\cot u \qquad \cot\left(\frac{\pi}{2}-u\right)=\tan u$$
$$\sec\left(\frac{\pi}{2}-u\right)=\csc u \qquad \csc\left(\frac{\pi}{2}-u\right)=\sec u$$

Examples: Use appropriate identities to simplify each expression.

$\sin(65^\circ)\sin(5^\circ)+\sin(-25^\circ)\sin(-85^\circ)$

$\sin(65^\circ)\sin(5^\circ)+(-\sin(25^\circ))(-\sin(85^\circ))$

$=\sin(65^\circ)\sin(5^\circ)+\sin(25^\circ)\sin(85^\circ)$

$=\cos(25^\circ)\cos(85^\circ)+\sin(25^\circ)\sin(85^\circ)$

$=\cos(25^\circ-85^\circ)=\cos(-60^\circ)=\boxed{1/2}$

$\cos(\pi/2-\alpha)\cos(-\alpha)-\sin(-\alpha)\sin(\alpha-\pi/2)$

$\sin\alpha\cos\alpha-(-\sin\alpha)(\sin(-(\frac{\pi}{2}-\alpha)))$

$\sin\alpha\cos\alpha+(\sin\alpha)(-\sin(\pi/2-\alpha))$

$\sin\alpha\cos\alpha-\sin\alpha\cos\alpha=\boxed{0}$

Sum and Difference Identities for Sine and Tangent

There are also sum and difference identities for sine and tangent:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Mnemonic for sine sum/difference: "Slytherin children are consistently sneaky."
1 pt. extra credit if you can come up with more helpful mnemonics for these identities.

Examples:

Find the exact value of $\sin(75^\circ)$.

$\sin(30^\circ+45^\circ)=$

$\sin 30^\circ\cos 45^\circ+\cos 30^\circ\sin 45^\circ=$

$(1/2)(\sqrt{2}/2)+(\sqrt{3}/2)(\sqrt{2}/2)=$

$\boxed{\dfrac{\sqrt{2}+\sqrt{6}}{4}}$

Find the exact value of $\sin(\pi/12)$.

$\sin(\pi/3-\pi/4)=$

$\sin\pi/3\cos\pi/4-\cos\pi/3\sin\pi/4$

$=\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{2}/2\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\boxed{\dfrac{\sqrt{6}-\sqrt{2}}{4}}$

Find the exact value of $\tan(15^\circ)$.

$\tan(60^\circ-45^\circ)=$

$\dfrac{\tan 60^\circ-\tan 45^\circ}{1+\tan 60^\circ\tan 45^\circ}=\dfrac{\sqrt{3}-1}{1+(\sqrt{3})(1)}$

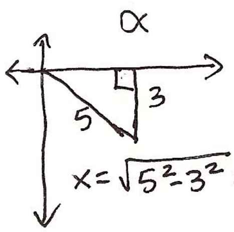
$=\boxed{\dfrac{\sqrt{3}-1}{1+\sqrt{3}}}$

Find the exact value of $\tan(5\pi/12)$.

$\tan(\pi/6+\pi/4)=\dfrac{\tan\pi/6+\tan\pi/4}{1-\tan\pi/6\tan\pi/4}$

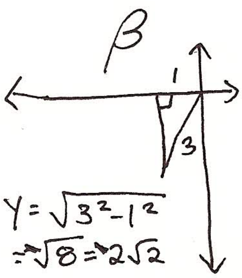
$=\dfrac{1/\sqrt{3}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}=\dfrac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}=\boxed{\dfrac{1+\sqrt{3}}{\sqrt{3}-1}}$

Find the exact value of $\sin(\alpha+\beta)$ if $\sin\alpha=-3/5$ and $\cos\beta=-1/3$ with α in Quadrant IV and β in Quadrant III.



$x=\sqrt{5^2-3^2}=4$

$\cos\alpha=4/5$



$y=\sqrt{3^2-1^2}$
$=\sqrt{8}=2\sqrt{2}$

$\sin\beta=-\frac{2\sqrt{2}}{3}$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$=(-3/5)(-1/3)+(4/5)(-2\sqrt{2}/3)$

$=\dfrac{3}{15}-\dfrac{8\sqrt{2}}{15}=\boxed{\dfrac{3-8\sqrt{2}}{15}}$

Double-Angle and Half-Angle Identities

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$$\sin(2x) = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$$

$$\cos(2x) = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$$

$$\tan(2x) = \tan(x+x) = \frac{\tan x + \tan x}{1-\tan x\tan x} = \frac{2\tan x}{1-\tan^2 x}$$

Example: Find $\sin\frac{2\pi}{3}$, $\cos\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ using double-angle identities.

$\sin(2\pi/3) = \sin(2\cdot\pi/3)$
$= 2\sin(\pi/3)\cos(\pi/3)$
$= 2(\sqrt{3}/2)(1/2) = \boxed{\sqrt{3}/2}$

$\cos(2\pi/3) = \cos(2\cdot\pi/3)$
$= \cos^2(\pi/3) - \sin^2(\pi/3)$
$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$
$= \frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = \boxed{-\frac{1}{2}}$

$\tan(2\pi/3) = \tan(2\cdot\pi/3)$
$= \frac{2\tan(\pi/3)}{1-\tan^2(\pi/3)}$
$= \frac{2\sqrt{3}}{1-(\sqrt{3})^2} = \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} = \boxed{-\sqrt{3}}$

Example: Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$ is an identity.

$\cos(3x) = \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x$
$= (\cos^2 x - \sin^2 x)(\cos x) - (2\sin x\cos x)(\sin x)$
$= \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x = \cos^3 x - 3\sin^2 x\cos x$

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the equations for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$2\cos^2 x - 1 = \cos(2x)$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

$$1 - 2\sin^2 x = \cos(2x)$$
$$-2\sin^2 x = \cos(2x) - 1$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

Since these equations work for any value of x, they also work if we replace x by $(x/2)$:

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

We can then use these formulas to derive formulas for $\tan\frac{x}{2}$:

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \tan\frac{x}{2}=\frac{\sin x}{1+\cos x} \qquad \tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$$

Examples: Use the half-angle identities to find the exact values of $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{8}$, and $\tan\frac{\pi}{8}$.

$$\sin(\pi/8)=\sin(\pi/4\div 2)$$
$$=\sqrt{\frac{1-\cos(\pi/4)}{2}}$$
$$=\sqrt{\frac{1-\sqrt{2}/2}{2}}$$
$$=\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

$$\cos(\pi/8)=\cos(\pi/4\div 2)$$
$$=\sqrt{\frac{1+\cos(\pi/4)}{2}}$$
$$=\sqrt{\frac{1+\sqrt{2}/2}{2}}$$
$$=\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}$$
$$=\boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$$

$$\tan(\pi/8)=\tan(\pi/4\div 2)$$
$$=\frac{1-\cos(\pi/4)}{\sin(\pi/4)}$$
$$=\frac{1-\frac{\sqrt{2}}{2}}{\sqrt{2}/2}$$
$$=\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{2}{\sqrt{2}}$$
$$=\frac{2}{\sqrt{2}}-1$$
$$=\boxed{\sqrt{2}-1}$$

Example: Prove that $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$ is an identity.

$$\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=$$
$$\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2$$
$$=\left(\frac{1-\cos x}{2}\right)\left(\frac{1+\cos x}{2}\right)$$
$$=\frac{1-\cos^2 x}{4}=\frac{\sin^2 x}{4}$$

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\cos(2\alpha)=-1/3$ and $\pi<2\alpha<3\pi/2$.

$$\cos(2\alpha)=2\cos^2\alpha-1$$
$$-1/3=2\cos^2\alpha-1$$
$$2/3=2\cos^2\alpha$$
$$1/3=\cos^2\alpha$$
$$\cos\alpha=-\sqrt{1/3}=\boxed{-\frac{1}{\sqrt{3}}}$$

$\pi/2<\alpha<3\pi/4 \leftarrow$ QII

$\sin\alpha$ & $\csc\alpha$ are positive

$$\sin^2\alpha+\cos^2\alpha=1$$
$$\sin^2\alpha+\left(-\frac{1}{\sqrt{3}}\right)^2=1$$
$$\sin^2\alpha+\frac{1}{3}=1$$
$$\sin^2\alpha=\frac{2}{3}$$
$$\sin\alpha=\boxed{\frac{\sqrt{2}}{\sqrt{3}}}$$

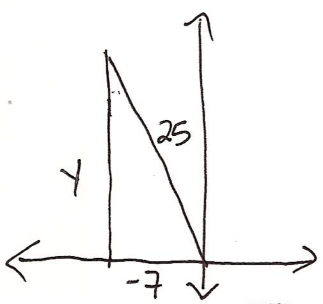
$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{\sqrt{2}}{\sqrt{3}}}{-\frac{1}{\sqrt{3}}}=\boxed{-\sqrt{2}}$$

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(\alpha/2)=4/5$ and $\pi/4<\alpha/2<\pi/2$.

$$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$(4/5)^2=\left(\pm\sqrt{\frac{1-\cos\alpha}{2}}\right)^2$$
$$\frac{16}{25}=\frac{1-\cos\alpha}{2}$$
$$\frac{32}{25}=1-\cos\alpha$$
$$\cos\alpha=1-\frac{32}{25}=\boxed{\frac{-7}{25}}$$

$\pi/2<\alpha<\pi \leftarrow$ QII

$\sin\alpha$ & $\csc\alpha$ are positive



$$y=\sqrt{25^2-7^2}=24$$

$$\boxed{\sin\alpha=\frac{24}{25}}$$
$$\boxed{\tan\alpha=-\frac{24}{7}}$$